time of interception) for given initial conditions are altered; conversely, the initial conditions are altered for given final conditions.

Control law (3.9) is valid for interception of a target in any orbit, be it circular, elliptical, parabolic, or hyperbolic.

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CRITERION OF EXISTENCE OF AN OPTIMAL CONTROL FOR A CLASS OF LINEAR STOCHASTIC SYSTEMS

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We consider a control system described by an nth order differential equation with random coefficients. Necessary and sufficient conditions of existence of a linear control stabilizing such a system in the mean square and conveying a minimum to the quadratic quality criterion are obtained. The problem of stabilization of a stochastic system in which the noise depends on the magnitude of the controlling force was also studied in [1].

1. Let a linear stochastic system be given, defined by the following nth order differential equation:

$$y^{(n)} + [a_1 + \xi_1^{\cdot}(t)]y^{(n-1)} + \ldots + [a_n + \xi_n^{\cdot}(t)]y = [b + \sigma\eta^{\cdot}(t)]u \qquad (1.1)$$

where

$$a_i = \text{const}, \qquad b_i = \text{const} \neq 0 \qquad (i = 1,2,\ldots,n)$$

u is a scalar control, $\xi_i^{\cdot}(t)$ are the Gaussian white noises with zero mathematical expectation which are, in general, interrelated in such a way that

$$M\xi_i^{\cdot}(t)\,\xi_j^{\cdot}(s) = 2a_{ij}\delta(t-s)$$

and $\eta^{\cdot}(t)$ is a white noise process independent of the set $\xi_1^{\cdot}(t),\ldots,\xi_n^{\cdot}(t)$. In addition

$$M\eta^{\cdot}(t) = 0, \qquad M\eta^{\cdot}(t)\eta^{\cdot}(s) = 2\delta(t-s)$$

Let us set

$$y = X_1, \qquad y' = X_2,\ldots,y^{(n-1)} = X_n$$

Then (1.1) can be assumed to represent a system of stochastic differential Ito equations (see e.g. [2])

$$dX_1 = X_2dt, \qquad dX_2 = X_3dt,\ldots,dX_{n-1} = X_ndt \qquad (1.2)$$

$$dX_n = \Big(-\sum_{i=1}^{n} a_iX_{n-i+1} + bu\Big)dt - \sum_{i,\,j=1}^{n} \alpha_{ij}X_{n-i+1}\,d\eta_j(t) + \sigma u\,d\eta(t)$$

where $\eta_1, \eta_2,\ldots,\eta_n$ and η denote mutually independent Gaussian Markov processes for which

$$M\eta^{\cdot}_i(t) = 0, \qquad M\eta_i^2(t) = 2t$$

and the matrix $\|\alpha_{ij}\|$ is obtained from the condition

$$\|\alpha_{ij}\|\,\|\alpha_{ji}\| = \|a_{ij}\|$$

Let us denote the solution of (1.2) with the initial condition $X_u^x(0) = x$ by $X_u^x(t)$. The generating operator of the process $X_u^x(t)$ has the form

$$L_u = \sum_{i=1}^{n} x_{i+1}\frac{\partial}{\partial x_i} + \left(bu - \sum_{i=1}^{n} a_i x_{n-i+1}\right)\frac{\partial}{\partial x_n} + \left(\sum_{i,\,j=1}^{n} a_{n-i+1\,n-j+1}x_i x_j + \sigma^2 u^2\right)\frac{\partial^2}{\partial x_n^2}$$

Let now P be a positive definite matrix. Our problem is to determine the law of control $u = u_0[x]$ under which the functional

$$J_x^P(u) = M\int_0^\infty [(PX_u^x(t),\ X_u^x(t)) + u^2(t)]\,dt \tag{1.3}$$

assumes its minimum value. Obviously the control $u = u[x]$ under which $J_x^P(u)$ is meaningful, brings the system (1.2) to asymptotic stability in the quadratic mean.

To solve our problem we have to determine the optimal Liapunov function $V_0(x) = (Cx,\ x)$ satisfying the Liapunov-Bellman equation [3 and 4]

$$L_{u_0}V_0 + (Px,\ x) + u_0^2 = \min_u\ [L_u V_0 + (Px,\ x) + u^2] = 0 \tag{1.4}$$

From (1.4) we obtain the following equation for the matrix $C = \|c_{ij}\|$ in the usual manner:

$$CB + B^*C - \frac{C\mathbf{b}\mathbf{b}^*C}{1 + 2\sigma^2 c_{nn}} + 2c_{nn}A + P = 0 \tag{1.5}$$

$$\mathbf{b} = \begin{Vmatrix} 0 \\ 0 \\ \\ 0 \\ b \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{Vmatrix}, \quad A = \|a_{n-i+1\,n-j+1}\|$$

where the superscript * denotes the operation of transposition. Moreover, if Eq. (1.4) has a positive definite solution $V_0(x)$, the optimal control has the form

$$u_0 = -\frac{b}{2(1 + 2\sigma^2\,\partial^2 V_0/\partial x_n^2)}\,\frac{\partial V_0}{\partial x_n}$$

To obtain a solution to the problem of optimal stabilization of the system (1.2) it is sufficient that there exists a positive definite matrix C satisfying Eq. (1.5). The optimal control $u = u_0[x]$ can be chosen in the form of a linear function of x. Further, using the method of consecutive approximations [5, 6] we can easily prove the following lemma.

Lemma 1.1. If a linear control $u = u[x]$ exists for the system (1.2) such that $J_x^{P^*}(u) < \infty$ for some positive definite matrix P^*, then a solution of (1.5) in the form of a unique positive definite matrix C exists for any positive definite matrix P.

2. Let us first assume that $\sigma \neq 0$ and, that the matrix A is positive definite. For $P = A/\sigma^2$, Eq. (1.5) has the form

$$CB + B^*C - \frac{C\mathbf{b}\mathbf{b}^*C}{1 + 2\sigma^2 c_{nn}} + \left(2c_{nn} + \frac{1}{\sigma^2}\right)A = 0 \tag{2.1}$$

which can obviously be written in the form

$$DB + B^*D - D\mathbf{b}_1\mathbf{b}_1^*D^* + A = 0 \tag{2.2}$$

$$D = \frac{C}{\sigma^{-2} + 2c_{nn}}, \qquad \mathbf{b}_1 = \mathbf{b}\sigma^{-1} \tag{2.3}$$

Since the system

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = b_1 u, \quad b_1 = b\sigma^{-1} \tag{2.4}$$

is fully controllable, Eq. (2.2) has a unique, positive definite solution $D_0 = \| d_{ij}^{(0)} \|$.

This together with (2.3) implies that the matrix

$$C_0 = \| c_{ij}^{(0)} \| = \frac{D_0}{\sigma^2 (1 - 2d_{nn}^{(0)})}$$

satisfies Eq. (2.1).

Let the condition $d_{nn}^{(0)} < 1/2$ hold. Then the matrix C_0 is positive definite and there exists a linear control

$$u_0 = -\frac{b}{2(1 + 2\sigma^2 c_{nn})} \frac{\partial V_0(x)}{\partial x_n}, \qquad V_0(x) = (C_0 x, x)$$

minimizing the functional (1.3).

Conversely, let the linear control $u = u[x]$ for which $J_x^P(u) < \infty$ exists for the system (1.2). Then by Lemma 1.1, Eq. (1.5) has a positive definite solution C_0, and obviously

$$D_0 = \frac{C_0}{\sigma^{-2} + 2c_{nn}^{(0)}}, \qquad d_{nn}^{(0)} = \frac{c_{nn}^{(0)}}{\sigma^{-2} + 2c_{nn}^{(0)}} < 1/2$$

Thus, when the matrix A is positive definite, an optimal control exists if and only if condition $d_{nn}^{(0)} < 1/2$ holds, where $d_{nn}^{(0)}$ denotes an element of the matrix $D_0 = \| d_{ij}^{(0)} \|$ representing the only positive definite solution of (2.2).

We shall now show how to express the coefficient $d_{nn}^{(0)}$ in terms of the parameters of the stochastic system (1.2).

To do this, we shall denote by $u_0 = \nu_1 y + \ldots + \nu_n y^{(n-1)}$ the control minimizing the functional

$$\int_0^\infty \left[\sum_{i,j=1}^n a_{n-i+1\, n-j+1} y^{(i-1)} y^{(j-1)} + u^2 \right] dt$$

on the solutions of (2.4) when $u = u_0$. It is easy to see that

$$d_{nn}^{(0)} = -\nu_n / b_1 \tag{2.5}$$

Further, we shall consider the polynomial

$$H(\lambda) = D(\lambda) D(-\lambda) + b_1^2 \sum_{i,j=1}^n a_{n-i+1\, n-j+1} (-1)^{i+1} \lambda^{i+j-2} \tag{2.6}$$

$$D(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n$$

Let us denote by α the sum of the roots of the equation $H(\lambda) = 0$ possessing positive real parts. As we know, there are exactly n such roots when the matrix A is positive definite (see e. g. [7]). From the results of [7, 8] it follows that $\nu_n = (a_1 - \alpha) b_1^{-1}$, therefore by (2.5) we have $d_{nn}^{(0)} = (\alpha - a_1) b_1^{-2}$.

Thus, when the matrix A is positive definite, an optimal control exists for the problem (1.2), (1.3) if and only if the following inequality holds:

$$2\sigma^2 (\alpha - a_1) < b^2 \tag{2.7}$$

Let us now assume that A is only nonnegative definite and that σ, as before, is not zero (in this case the equation $H(\lambda) = 0$ can have purely imaginary roots). We supplement the coefficients a_i of the system (1.2) with white noises of low dispersion ε, independent of each other and of η_i and η. Subsequently proceeding to the limit as $\varepsilon \to 0$ (as in [9]), we find that the inequality (2.7) represents the sufficient condition, and the relation $2\sigma^2(\alpha - a_1) \leqslant b^2$ the necessary condition of existence of an optimal linear control, solving the problem in question.

It can be shown that, in fact, the equality sign in the latter relation must be omitted.

This can be easily verified for the cases $n = 1$ and 2.

Thus we have the following theorem.

Theorem 2.1. Let $\sigma \neq 0$ and $\lambda_1, \ldots, \lambda_k$, $k \leqslant n$ be roots of the equation $H(\lambda) = 0$ possessing positive real parts. If the inequality

$$2\sigma^2 \left(\sum_{i=1}^{k} \lambda_i - a_1 \right) < b^2 \tag{2.8}$$

holds, the problem of optimal stabilization of the system (1.2) with the quality criterion (1.3) has a solution in the class of linear controls; but if the inequality

$$2\sigma^2 \left(\sum_{i=1}^{k} \lambda_i - a_1 \right) > b^2$$

holds, then no such solution exists.

However, when the matrix A is positive definite ($k = n$), the inequality (2.8) will not only be sufficient but will also become necessary for the existence of an optimal linear control stabilizing the system (1.2).

Let us now set $\sigma = 0$. Using the fact that $H(\lambda) = 0$ contains only even powers of λ, we can easily establish that $|\lambda_i|^2 = O(\sigma^{-2})$ when $\sigma \to 0$. Consequently, for any parameters a_i, $b \neq 0$ and $a_{i,}$ there exists a sufficiently small $\sigma = \sigma^*$ for which the inequatity (2.7) holds, i.e. the system (1.2) is asymptotically stable in the quadratic mean when $\sigma = \sigma^*$. This, together with Tneorem 5.1 of [10] and Lemma 1.1 yields at once the following theorem.

Theorem 2.2. Let $\sigma = 0$. Then for any parameters a_i, $b \neq 0$ and a_{ij} there exists a linear control $u = u_0[x]$ minimizing the functional (1.3) on the solutions of the system (1.2) with $u = u_0[x]$.

3. Let us consider a particular case of the following stochastic system:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = (b + \sigma\eta^{\cdot}(t))u$$

For this system we nave

$$H(\lambda) = D(\lambda) D(-\lambda)$$

Let $\mu_1, \ldots, \mu_l$ and $\nu_1, \ldots, \nu_m$, $l + m \leqslant n$ be the roots of the polynomial $D(\lambda) = 0$ with positive and negative real parts, respectively. Then we obviously have

$$\alpha = (\mu_1 + \ldots + \mu_l - \nu_1 - \ldots - \nu_m)$$

Further, $a_1 = -(\mu_1 + \ldots + \mu_l + \nu_1 + \ldots + \nu_m)$, therefore from Theorem 2.1 it follows that the condition of existence of an optimal control is:

$$4\sigma^2 \sum_{i=1}^{l} \mu_i < b^2$$

Finally we note that the first and second order conditions of existence of an optimal control for the system (1.2) are respectively:

$$\sqrt{a_1^2 + (b/\sigma)^2 a_{11}} - a_1 < {}^1/_2 (b/\sigma)^2 \quad \text{for } n = 1$$

$$\sqrt{a_1^2 + (b/\sigma)^2 a_{22} - 2a_2 + 2\sqrt{a_2^2 + (b/\sigma)^2 a_{11}}} - a_1 < {}^1/_2 (b/\sigma)^2 \quad \text{for } n = 2$$

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ON A CRITERION OF STABILITY OF SOLUTIONS OF AN N-TH ORDER LINEAR DIFFERENTIAL EQUATION

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Using the methods of the theory of cones we establish a sufficient condition of stability of solutions of an n-th order linear differential equation.

Let us consider the following linear differential equation:

$$\frac{d^n x}{dt^n} + p_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + p_n(t)\,x = 0 \tag{1}$$

where $p_i(t)$ $(i = 1, \ldots, n;\ t_0 \leqslant t < \infty)$ are continuous functions. We shall indicate one criterion of the stability of solutions of (1) in terms of its characteristic polynomial

$$P(t, \lambda) = \lambda^n + p_1(t)\lambda^{n-1} + \ldots + p_n(t) \tag{2}$$

We will use certain concepts of the theory of cones [1, 2].

Let us write Eq. (1) in the form of a first order equation in an n-dimensional Euclidean space R^n